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RESEARCH ARTICLE

A New Generalized Fractional Riemann-Liouville Derivative Operator Using Wright Function

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Abstract

Numerous extensions of beta function have been investigated by various researchers .In this paper we develop new generalized beta function and obtain integral representations and summation formulas based on it. We also establish some integral representations of new generalised hypergeometric and confluent hypergeometric functions. Further we introduce new generalized fractional Riemann Liouville differ-integral operator and discuss the properties like Mellin transform etc.

Keywords: ψ-Hypergeometric Function, ψ-Generalized Beta Function, Mellin Transform, ψ-Riemann-Liouville Fractional Derivative Operator.

AMS Subject Classification: 33C05, 26A33,33C65, 33B15.

Introduction

The beta function has great importantance in calculus and analysis due to its close connection to the gamma function, which is itself a generalization of the factorial function. Many complex integrals can be reduced to expressions involving the beta function. Various studies show the work of extensions of beta functions by several researchers in past years (Ozergin *et al.*, 2011; Parmar *et al.*, 2017; Atash *et al.*, 2018, Mubeen *et al.*, 2017).

Chaudhry *et al.* (1997), have obtained generalized beta function as:

$$\mathcal{B}_p(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}exp(\frac{-p}{u(1-u)})dt, \quad (1)$$

where $R(p) > 0; R(x) > 0, R(y) > 0$.

Chaudhary *et al.* (2004) have also established the generalization of the Gauss-hypergeometric function and confluent hypergeometric function as follow:

$$\mathcal{F}_p(\eta,\zeta;\varrho;z) = \sum_{k=0}^{\infty}(\eta)_k \frac{\mathcal{B}_p(\zeta+k\,,\varrho-\zeta)}{\mathcal{B}(\zeta,\varrho-\zeta)}\frac{z^k}{k!}, \quad (2)$$

where $\mathcal{R}(\varrho) > \mathcal{R}(\zeta) > 0, \mathcal{R}(\eta) > 0, p \geq 0; |z| < 1$.
and

$$\Phi_p(\zeta;\varrho;z) = \sum_{k=0}^{\infty} \frac{\mathcal{B}_p(\zeta+k\,,\varrho-\zeta)}{\mathcal{B}(\zeta,\varrho-\zeta)}\frac{z^k}{k!}, \quad (3)$$

$p \geq 0; |z| < 1; \mathcal{R}(\varrho) > \mathcal{R}(\zeta) > 0.$

Here, $(c)_\lambda$ represents Pochammer's symbol defined by

$$(c)_\lambda = \frac{\Gamma(c+\lambda)}{\Gamma(c)}. \quad (4)$$

Integral representations of these functions are given by these following equations as:

$$\mathcal{F}_p(\eta,\zeta;\varrho;z) = \frac{1}{\mathcal{B}(\zeta,\varrho-\zeta)}\int_0^1 t^{\zeta-1}(1-t)^{\varrho-\zeta-1} \times (1-zt)^{-\eta}exp\Big(\frac{-p}{t(1-t)}\Big)dt, \quad (5)$$

$\mathcal{R}(\varrho) > \mathcal{R}(\zeta) > 0; p \geq 0; |arg(1-z)| < \pi.$

and

$$\Phi_p(\zeta;\varrho;z) = \frac{1}{\mathcal{B}(\zeta,\varrho-\zeta)}\int_0^1 t^{\zeta-1}(1-t)^{\varrho-\gamma-1} exp\Big(zt - \frac{-p}{t(1-t)}\Big)dt, \quad (6)$$

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$p \geq 0$ and$\mathcal{R}(\varrho) > \mathcal{R}(\zeta) > 0$.

After putting $p = 0$ above results reduces due to the well known beta-function.

Choi *et al.*(2014) obtained the extended beta function as

$$\mathcal{B}_{p,q}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} exp\Big(\frac{-p}{u} - \frac{q}{1-u}\Big)dt, \quad (7)$$

$\mathcal{R}(p), \mathcal{R}(q) > 0; \mathcal{R}(x) > 0,\ \mathcal{R}(y) > 0$.

Note:

(i). When $p = q$, equation (7) reduces to equation (1), which is given by Chaudhry *et. al.* (1997)

(ii). When $p = q = 0$, equation (7) reduces to the original beta function.

Shahid Mubeen *et al.* (2017) introduced generalization of extended beta function as

$$\mathcal{B}_{p,q}^{\sigma,\tau}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} {}_1F_1\Big(\sigma,\tau;\frac{-p}{u}\Big) {}_1F_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du, \quad (8)$$

Where $\mathcal{R}(p) > 0, \mathcal{R}(q) > 0, \mathcal{R}(\sigma) > 0, \mathcal{R}(\tau) > 1, \mathcal{R}(x) > 0, \mathcal{R}(y) > 0$.

Note:

(i) When $\sigma = \tau$, $p \neq q$, then equation (8) reduces to the extended beta function given by choi. *et.al.*

(ii) When $\sigma = \tau$, $p = q$, then equation (8) reduces to the extended beta function given by Chaudhary *et.al.*

(iii) When $\sigma = \tau$, $p = q = 0$, then equation (8) reduces to the classical beta function.

Ata (Ata, 2018) obtained the following generalization gamma and beta functions as

$$^{\psi}\Gamma_p^{(\sigma,\tau)}(x) = \int_0^\infty u^{x-1} {}_1\Psi_1\Big(\sigma,\tau;-u-\frac{p}{u}\Big)du, \quad (9)$$

where $\sigma, \tau \in C, p > 0; \mathcal{R}(x) > 0, \mathcal{R}(\sigma) > 0,\ (\tau) > 1$ and ${}_1\Psi_1(\sigma,\tau;z)$ is the Wright function defined by

$${}_1\Psi_1(\sigma,\tau;z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\sigma k + \tau)}\frac{z^k}{k!},$$

and

$$^{\psi}\mathcal{B}_p^{(\sigma,\tau)}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u(1-u)}\Big)du, \quad (10)$$

Where $\mathcal{R}(p) > 0, \mathcal{R}(\tau) > 1, \mathcal{R}(\sigma) > 0, \mathcal{R}(x) > 0, \mathcal{R}(y) > 0$.

For $p = 0, \sigma = 0, \tau = 2$ equation (8) reduces to the classical gamma function and for $p = 0, \tau = 2$ equation (10) reduces to the classical beta-function.

And ψ- Gauss hypergeometric function and ψ- Confluent hypergeometric function is given by

$$^{\psi}\mathcal{F}_p^{(\sigma,\tau)}(\eta,\zeta;\varrho;z) = \sum_{k=0}^\infty (\eta)_k \frac{^{\psi}\mathcal{B}_p^{(\sigma,\tau)}(\zeta+k,\varrho-\zeta)}{\mathcal{B}(\zeta,\varrho-\zeta)}\frac{z^n}{n!}, \quad (11)$$

and

$$^{\psi}\Phi_p^{(\sigma,\tau)}(\zeta;\varrho;z) = \sum_{k=0}^\infty \frac{^{\psi}\mathcal{B}_p^{(\sigma,\tau)}(\zeta+k,\varrho-\zeta)}{\mathcal{B}(\zeta,\varrho-\zeta)}\frac{z^n}{k!}. \quad (12)$$

Obviously, If we put $p = 0, \tau = 2$ equations (11) and (12) reduce to the original Gauss hypergeometric and confluent hypergeometric function respectively.

The Integral representation of the above (11) and (12) is given by

$$^{\psi}\mathcal{F}_p^{(\sigma,\tau)}(\eta,\zeta;\varrho;z) = \frac{1}{\mathcal{B}(\zeta,\varrho-\zeta)}\int_0^1 u^{\zeta-1}(1-u)^{\varrho-\zeta-1} {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u(1-u)}\Big)(1-zu)^{-\eta}du, \quad (13)$$

where $\mathcal{R}(\varrho) > \mathcal{R}(\zeta) > 0, \mathcal{R}(p) > 0, |arg(1-z)| < \pi$.

$$^{\psi}\Phi_p^{(\sigma,\tau)}(\zeta;\varrho;z) = \frac{1}{\mathcal{B}(\zeta,\varrho-\zeta)}\int_0^1 u^{\zeta-1}(1-u)^{\varrho-\zeta-1} {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u(1-u)}\Big)du, \quad (14)$$

where $\mathcal{R}(\varrho) > \mathcal{R}(\zeta) > 0, \mathcal{R}(p) > 0, |arg(1-z)| < \pi$.

The whole work is divided into sections. Section 2 is concern about the results related to new generalized beta function. Section 3 focuses on generalization of Riemann-Liouville fractional differ- integral operator and related results.

New Generalized Beta Function Defined by Wright Function

In this section we introduce a (p,q)- extended beta function by using the product of two wright functions,which reduces to some known extended beta functions as well as classical beta function. We begin by defining the following generalization of beta function as-

$$^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big)\, {}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du, \quad (15)$$

Where $\sigma\in(0,2),\tau\in R,\tau>1,\mathcal{R}(x)>0,\mathcal{R}(y)>0,\mathcal{R}(p)>0,\mathcal{R}(q)>0$ and ${}_1\Psi_1(\sigma,\tau;z)$ is the Wright function defined by

$${}_1\Psi_1(\sigma,\tau;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\sigma k+\tau)}\frac{z^k}{k!}.$$

Special cases:

(i) When $\sigma=0,\tau=2,p\neq q$, then equation (15) convert to the extended beta function given by Choi *et.al*.

(ii) When $\sigma=0,\tau=2,p=q$, then equation (15) convert to the extended beta function given by Chaudhary *et.al*.

(iii) When $\sigma=0,\tau=2,p=q=0$ then equation (15) reduces to the classical beta function.

Now we introduce a generalized Gauss hypergeometric and Confluent hypergeometric function as

$$^{\psi}\mathcal{F}_{p,q}^{(\sigma,\tau)}(\eta,\zeta;\varrho;z)=\sum_{n=0}^{\infty}(\eta)_n\frac{^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(\zeta+n,\varrho-\zeta)}{\mathcal{B}(\zeta,\varrho-\zeta)}\frac{z^n}{n!}, \quad (16)$$

and

$$^{\psi}\Phi_{p,q}^{(\sigma,\tau)}(\zeta;\varrho;z)=\sum_{n=0}^{\infty}\frac{^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(\zeta+n,\varrho-\zeta)}{\mathcal{B}(\zeta,\varrho-\zeta)}\frac{z^n}{n!}. \quad (17)$$

Obviously, for $\sigma=0,\tau=2,p=q=0$ equations (16) and (17) reduce to the original Gauss hypergeometric function and confluent hypergeometric function respectively.

Further we define integral representation of Generalized Gauss hypergeometric and Confluent hypergeometric function as

$$^{\psi}\mathcal{F}_{p,q}^{(\sigma,\tau)}(\eta,\zeta;\varrho;z)=\frac{1}{\mathcal{B}(\zeta,\varrho-\zeta)}\int_0^1 u^{\zeta-1}(1-u)^{\varrho-\zeta-1} \times {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big){}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)(1-zu)^{-\eta}du, \quad (18)$$

where $\mathcal{R}(p)>0,\mathcal{R}(q)>0,\mathcal{R}(\varrho)>\mathcal{R}(\zeta)>0,|arg(1-z)|<\pi,\sigma\in(0,2),\tau\in R,\tau>1$.

and

$$^{\psi}\Phi_{p,q}^{(\sigma,\tau)}(\zeta;\varrho;z)=\frac{1}{\mathcal{B}(\zeta,\varrho-\zeta)}\int_0^1 u^{\zeta-1}(1-u)^{\varrho-\zeta-1} \times {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big){}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du, \quad (19)$$

where $\mathcal{R}(p)>0,\mathcal{R}(q)>0,\mathcal{R}(\varrho)>\mathcal{R}(\zeta)>0,|arg(1-z)|<\pi,\sigma\in(0,2),\tau\in R,\tau>1$.

Theorem 2.1. *The following integral representation holds*

$$^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y)=2\int_0^{\pi/2} cos^{2x-1}\Theta sin^{2y-1}\Theta \times {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{cos^2\Theta}\Big){}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{sin^2\Theta}\Big)du, \quad (20)$$

where $\mathcal{R}(p)>0,\mathcal{R}(q)>0,\mathcal{R}(x)>0,\mathcal{R}(y)>0,\sigma\in(0,2),\tau\in R,\tau>1$.

Proof. Put $t=cos^2\Theta$ in equation (15) we get the required result. □

Theorem 2.2. *For* $\mathcal{R}(x)>0,\mathcal{R}(y)>0,\mathcal{R}(p)>0,\mathcal{R}(q)>0,\sigma\in(0,2),\tau\in R,\tau>1$, *following formula holds:*

$$^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y)=2\int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}-p\Big) \times {}_1\Psi_1\Big(\sigma,\tau;-q-qu\Big)du. \quad (21)$$

Proof. Put $t=\frac{u}{1+u}$ in equation (15) we get the required result. □

Theorem 2.3. *Let* $\mathcal{R}(x)>0,\mathcal{R}(y)>0,\mathcal{R}(p)>0,\mathcal{R}(q)>0,\sigma\in(0,2),\tau\in R,\tau>1$,*then following formula holds:*

$$^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y)=(d-a)^{1-x-y}\int_{-1}^1(u-a)^{x-1}(d-u)^{y-1} \times {}_1\Psi_1\Big(\sigma,\tau;\frac{-p(d-a)}{u-a}\Big){}_1\Psi_1\Big(\sigma,\tau;\frac{-q(d-a)}{d-u}\Big)du. \quad (22)$$

Proof. Put $t=\frac{u-a}{d-a}$ in equation (15) we get the required result. □

Theorem 2.4. *For* $\mathcal{R}(x)>0,\mathcal{R}(y)>0,\mathcal{R}(p)>0,\mathcal{R}(q)>0,\sigma\in(0,2),\tau\in R,\tau>1,n\in N$, *following formula holds*

$$ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y)=\sum_{r=0}^{n}{}^{n}C_r\ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x+r,y+n-r). \tag{23}$$

Proof. By equation (15) we have

$$\begin{aligned} {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y) &= \int_0^1 u^{x-1}(1-u)^{y-1}{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big)\\ &\quad\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du \qquad (24)\\ &= \int_0^1 [u^{x}(1-u)^{y-1}+u^{x-1}(1-u)^{y}]\\ &\quad\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big){}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du\\ &= \int_0^1 [u^{x+1-1}(1-u)^{y-1}+u^{x-1}\\ &\quad\times(1-u)^{y+1-1}]{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big){}_1\\ &\quad\times\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du\\ &={}^{\psi}\ \mathcal{B}_{p,q}^{(\sigma,\tau)}(x+1,y){}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y+1)\\ &=\sum_{r=0}^{1}{}^{1}C_r\ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x+r,y+1-r). \end{aligned}$$

Now right and side of equation (24) can be expanded as

$$\begin{aligned} {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y) &= \int_0^1 [u^{x-1}(1-u)^{y+1}+2u^{x}(1-u)^{y}\\ &\quad+u^{x+1}(1-u)^{y-1}]{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big)\\ &\quad\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du\\ &= \int_0^1 [u^{x-1}(1-u)^{y+2-1}+2u^{x+1-1}\\ &\quad\times(1-u)^{y+1-1}+u^{x+2-1}(1-u)^{y-1}]{}_1\\ &\qquad\times\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big){}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du\\ &={}^{\psi}\ \mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y+2)2{}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x+1,y+1)\\ &\qquad+\mathcal{B}_{p,q}^{(\sigma,\tau)}(x+2,y). \qquad (25) \end{aligned}$$

The same argument is repeatedly applied to the right side of equation (24), upto n times, we get desired result

Theorem 2.5. *For* $R(p)>0,R(q)>0,Re(x)>0Re(y)>0,\sigma\in(0,2),\tau\in R,\tau>1$, *following formula holds*

$$ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,1-y)=\sum_{k=0}^{\infty}\frac{(y)_k}{k!}\ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x+k,1). \tag{26}$$

Proof.

$$\begin{aligned} {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,1-y) &= \int_0^1 u^{x-1}(1-u)^{-y}\ {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big)\\ &\quad\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du. \qquad (27) \end{aligned}$$

By using the binomial theorem, we get

$$(1-u)^{-y}=\sum_{k=0}^{\infty}(y)_k\frac{u^k}{k!}. \tag{28}$$

using equation (28) in equation(27) we get the required result.

Theorem 2.6. *For* $R(p)>0,R(q)>0,Re(x)>0Re(y)>0,\sigma\in(0,2),\tau\in R,\tau>1$, *following formula holds*

$$ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y)=\sum_{k=0}^{\infty}\ {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x+k,y+1). \tag{29}$$

Proof.

$$\begin{aligned} {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(x,y) &= \int_0^1 u^{x-1}(1-u)^{y-1}\ {}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big)\\ &\quad\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)du. \qquad (30) \end{aligned}$$

By using the binomial theorem, we get

$$(1-u)^{y-1}=(1-u)^{y}\sum_{k=0}^{\infty}u^k. \tag{31}$$

using equation (31) in equation(30) we get the required result.

Generalized Riemann Liouville fractional derivative operator

The Classical Riemann -Liouville (R-L)fractional integral operator of order is defined by

$$I_z^{\mu}f(z)=\frac{1}{\Gamma(\mu}\int_0^z(z-u)^{\mu-1}f(u)du,\qquad \mathcal{R}(\mu)>0. \tag{32}$$

and the Riemann -Liouville (R-L)fractional derivative operator of order μ is defined by

$$D_z^{\mu}f(z)=\frac{1}{\Gamma(k-\mu)}\frac{d^k}{dz^k}\int_0^z(z-u)^{k-\mu-1}f(u)du \tag{33}$$

$$= \frac{d^k}{dz^k}[I_z^{k-\mu} f(z)],$$

where $\mathcal{R}(\mu) > 0, k = \mathcal{R}(\mu) + 1, z > 0$.

Many researchers (Srivastava *et al.*,2013; Agarwal *et al.*,2017; Bohner *et al.*,2018; Shadab *et al.*,2018) studied the many generalizations of fractional integral and derivative operators.

Extension of Riemann-Liouville fractional integral of order μ is defined in Özarslan and Özergin, 2010 by

$$I_z^{\mu,p} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z f(u)(z-u)^{\mu-1} exp\Big(\frac{-pz^2}{u(z-u)} du\Big) du, \tag{34}$$

where $\mathcal{R}(\mu) > 0, \mathcal{R}(p) > 0$.

and the Riemann -Liouville (R-L)fractional derivative operator of order μ is defined by

$$D_z^{\mu,p} f(z) = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dz^k} \int_0^z f(u)(z-u)^{k-\mu-1} \times exp\Big(\frac{-pz^2}{u(z-u)} du\Big) du = \frac{d^k}{dz^k}[I_z^{k-\mu,p} f(z)], \tag{35}$$

where $\mathcal{R}(\mu) > 0, \mathcal{R}(p) > 0, k = \mathcal{R}(\mu) + 1, z > 0$.

Baleanu *et al.* (2015) extended the fractional integral and derivative as follows

$$I_z^{\mu}\{f(z); p, q\} = \frac{1}{\Gamma(\mu)} \int_0^z f(u)(z-u)^{\mu-1} \times exp\Big(\frac{-pz}{u} - \frac{qz}{z-u} du\Big) du, \tag{36}$$

where $\mathcal{R}(\mu) > 0, min(\mathcal{R}(p), \mathcal{R}(q)) > 0$ and

$$D_z^{\mu}\{f(z); p, q\} = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dz^k} \int_0^z f(u)(z-u)^{k-\mu-1} \times exp\Big(\frac{-pz}{u} - \frac{qz}{z-u} du\Big) du = \frac{d^k}{dz^k}[I_z^{k-\mu}\{f(z); p, q\}], \tag{37}$$

where $\mathcal{R}(\nu) > 0, min(\mathcal{R}(p), \mathcal{R}(q)) > 0, k = \mathcal{R}(\nu) + 1, z > 0$.

Rahman *et al.* (2019) give further extension of fractional integral and derivative as follows

$$I_z^{\mu}\{f(z); p, q; \lambda, \rho\} = \frac{1}{\Gamma(\mu)} \int_0^z f(u)(z-u)^{\mu-1} \times {}_1F_1\Big(\lambda, \rho, \frac{-pz}{u}\Big) {}_1F_1\Big(\lambda, \rho, \frac{-qz}{z-u}\Big), \tag{38}$$

where $\mathcal{R}(\mu) > 0, min(\mathcal{R}(p), \mathcal{R}(q)) > 0, \lambda \in C, \rho \in C \backslash Z_0^-$.

and

$$D_z^{\mu}\{f(z); p, q; \lambda, \rho\} = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dz^k} \int_0^z f(u) \times (z-u)^{k-\mu-1} {}_1F_1\Big(\lambda, \rho, \frac{-pz}{u}\Big) \times {}_1F_1\Big(\lambda, \rho, \frac{-qz}{z-u}\Big) du = \frac{d^k}{dz^k}[I_z^{k-\mu} f(z)]. \tag{39}$$

where $\mathcal{R}(\mu) > 0, min(\mathcal{R}(p), \mathcal{R}(q)) > 0, k = \mathcal{R}(\mu) + 1, z > 0, \lambda \in C, \rho \in C \backslash Z_0^-$.

In this section, we define new extension of Riemann Liouville fractional integral and derivative operator using Wright function as:

$${}^{\psi}I_z^{\mu}\{f(z); p, q; \sigma, \tau\} = \frac{1}{\Gamma(\mu)} \int_0^z f(u)(z-u)^{\mu-1} \times {}_1\psi_1\Big(\sigma, \tau, \frac{-pz}{u}\Big) {}_1\psi_1\Big(\sigma, \tau, \frac{-qz}{z-u}\Big), \tag{40}$$

where $min(\mathcal{R}(p), \mathcal{R}(q)) > 0, \mathcal{R}(\mu) > 0, \sigma \in C, \tau \in C \backslash Z_0^-$.

and

$${}^{\psi}D_z^{\mu}\{f(z); p, q; \sigma, \tau\} = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dz^k} \int_0^z f(u) \times (z-u)^{k-\mu-1} {}_1\psi_1\Big(\sigma, \tau, \frac{-pz}{u}\Big) {}_1 \times \psi_1\Big(\sigma, \tau, \frac{-qz}{z-u}\Big) du = \frac{d^k}{dz^k}[I_z^{k-\mu} f(z)], \tag{41}$$

where $min(\mathcal{R}(p), \mathcal{R}(q)) > 0, \mathcal{R}(\mu) > 0, k = \mathcal{R}(\mu) + 1, z > 0, \sigma \in C, \tau \in C \backslash Z_0^-$.

Special cases:

(i) When $\sigma = 0, \tau = 2, p = q$, then equation (40) and (41) reduces to the Riemann Liouville fractional integral and derivative operator given by Özarslan *et.al.*

(ii) When $\sigma = 0, \tau = 2, p = q = 0$ then equation (40) and (41) reduces to the well known Riemann Liouville fractional integral and derivative operator function.

Applications

Theorem 4.1. *The following result holds true*

$$I_z^{\mu}\{z^{\nu}; p, q; \sigma, \tau\} = \frac{z^{\nu+\mu}}{\Gamma(\mu)} \, {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(\nu+1, \mu), \quad (42)$$

where $\mathcal{R}(\mu) > 0, \mathcal{R}(\nu) > -1, min(\mathcal{R}(p), \mathcal{R}(q)) > 0, k = \mathcal{R}(\mu)+1, z > 0, \sigma \in (0,2), \tau \in R, \tau > 1.$

Proof. put $f(u) = u^{\nu}$ in the equation (40) we get

$$I_z^{\mu}\{z^{\nu}; p, q; \sigma, \tau\} = \frac{1}{\Gamma(\mu)} \int_0^z u^{\mu}(z-u)^{\mu-1}{}_1 \times \psi_1\Big(\sigma, \tau, \frac{-pz}{u}\Big){}_1\psi_1\Big(\sigma, \tau, \frac{-qz}{z-u}\Big).$$

let $u = zs$

$du = zds$

$$I_z^{\mu}\{z^{\nu}; p, q; \sigma, \tau\} = \frac{1}{\Gamma(\mu)} \int_0^z (zs)^{\mu}(z-zs)^{\mu-1} \times {}_1\psi_1\Big(\sigma, \tau, \frac{-pz}{zs}\Big){}_1\psi_1\Big(\sigma, \tau, \frac{-qz}{z-zs}\Big)$$

$$= \frac{1}{\Gamma(\mu)} \int_0^z z^{\nu+\mu}u^{\mu}(1-s)^{\mu-1} \times {}_1\psi_1\Big(\sigma, \tau, \frac{-p}{s}\Big){}_1\psi_1\Big(\sigma, \tau, \frac{-q}{1-s}\Big)$$

$$= \frac{z^{\nu+\mu}}{\Gamma(\mu)} \, {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(\nu+1, \mu).$$

Theorem 4.2. *Suppose the function* $\mathfrak{F}(z)$ *be an analytic in* $|z| < \varepsilon$ *defined by* $\mathfrak{F}(z) = \sum_{r=0}^{\infty} c_r z^r$ *and* $R(\nu) < 0, \sigma \in (0,2), \tau \in R, \tau > 1$, *then*

$${}^{\psi}I_{z,p,q}^{\mu,\sigma,\tau}\{\mathfrak{F}(z)\} = \sum_{r=0}^{\infty} c_r \; {}^{\psi}I_{z,p,q}^{\mu,\sigma,\tau}\{z^r\}$$

$$= \frac{1}{\Gamma(-\mu)} \sum_{r=0}^{\infty} c_r \, {}^{\psi}\mathcal{B}_{p,q}^{(\sigma,\tau)}(r+1, -\mu) z^{r-\mu} \quad (43)$$

Proof. By putting $\mathfrak{F}(z) = \sum_{r=0}^{\infty} c_r z^r$ in (40),we have

$${}^{\psi}I_{z,p}^{\mu,\sigma,\tau}\{\mathcal{F}(z)\} = \frac{1}{\Gamma(-\mu)} \int_0^z \sum_{r=0}^{\infty} c_r u^r (z-u)^{-\mu-1} \times {}_1\Psi_1\Big(\sigma, \tau; \frac{-pz}{u}\Big){}_1\Psi_1\Big(\sigma, \tau; \frac{-qz}{(z-u)}\Big)du.$$

After changing order of integration and summation, we get

$${}^{\psi}I_{z,p,q}^{\mu,\sigma,\tau}\{\mathfrak{F}(z)\} = \sum_{r=0}^{\infty} c_r \; {}^{\psi}I_{z,p,q}^{\mu,\sigma,\tau}\{z^r\}.$$

Finally, using equation (42) in above result, we get desired result.

Theorem 4.3. *Let* $R(\mu) > R(\eta) > 0, \mathcal{R}(b) > 0$ *and* $|z| < 1, \tau \in R, \sigma \in (0,2), \tau > 1$,*then*

$${}^{\psi}I_{z,p,q}^{\mu-\eta,\sigma,\tau}\{z^{\eta-1}(1-z)^{-b}\} = \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \, {}^{\psi}\mathcal{F}_{p,q}^{(\sigma,\tau)}(b, \eta; \mu; z). \quad (44)$$

Proof. Using the equation(40), we have

$${}^{\psi}I_{z,p,q}^{\eta-\mu,\sigma,\tau}\{z^{\eta-1}(1-z)^{-b}\}$$

$$= \frac{1}{\Gamma(\mu-\eta)} \int_0^z t^{\eta-1}(1-t)^{-b}(z-t)^{\mu-\eta-1} \times {}_1\Psi_1\Big(\sigma, \tau; \frac{-pz}{t}\Big){}_1\Psi_1\Big(\sigma, \tau; \frac{-qz}{z-t}\Big)dt$$

$$= \frac{1}{\Gamma(\mu-\eta)} z^{\mu-\eta-1} \int_0^z t^{\eta-1}(1-t)^{-b} \times (1-\frac{t}{z})^{\mu-\eta-1}{}_1\Psi_1\Big(\sigma, \tau; \frac{-pz}{t}\Big) \times {}_1\Psi_1\Big(\sigma, \tau; \frac{-qz}{z-t}\Big)dt,$$

put t=uz

$$= \frac{1}{\Gamma(\mu-\eta)} z^{\mu-1} \int_0^1 u^{\eta-1}(1-uz)^{-b} \times (1-u)^{\mu-\eta-1}{}_1\Psi_1\Big(\sigma, \tau; \frac{-p}{u}\Big) \times {}_1\Psi_1\Big(\sigma, \tau; \frac{-q}{z-u}\Big)du,$$

by (18)

$$= \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} \, {}^{\psi}\mathcal{F}_{p,q}^{(\sigma,\tau)}(b, \eta; \mu; z).$$

Theorem 4.4. *Let* $\mathcal{R}(\lambda) > -1, \mathcal{R}(\mu) < 0, \mathcal{R}(r) > 0\mathcal{R}(s) > 0, \tau \in R, \sigma \in (0,2), \tau > 1$,, *then*

$$\mathfrak{M}[{}^{\psi}I_{z,p,q}^{\mu,\sigma,\tau}\{z^{\lambda}\}; p \to r, q \to s] = \frac{z^{\lambda+\nu}}{\Gamma(\mu)}\mathcal{B}(\lambda+r+1, s+\mu) \, {}^{\psi}\Gamma^{(\sigma,\tau)}(r) \, {}^{\psi}\Gamma^{(\sigma,\tau)}(s). \quad (45)$$

Proof. Applying the definition of Mellin transform we obtain

$$\mathfrak{M}[{}^{\psi}I^{\mu,\sigma,\tau}_{z,p,q}\{z^{\lambda}\};p\to r,q\to s]$$
$$=\frac{1}{\Gamma(\mu)}\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\Big\{\int_0^z t^{\lambda}$$
$$\times(z-t)^{\mu-1}{}_1\Psi_1\Big(\sigma,\tau;\frac{-pz}{t}\Big)$$
$$\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-qz}{z-t}\Big)dt\Big\}dpdq$$
$$=\frac{z^{\mu-1}}{\Gamma(\mu)}\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\Big\{\int_0^z t^{\lambda}$$
$$\times(1-\frac{t}{z})^{\mu-1}{}_1\Psi_1\Big(\sigma,\tau;\frac{-pz}{t}\Big)$$
$$\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-qz}{z-t}\Big)dt\Big\}dpdq,$$

put t=uz

$$=\frac{z^{\lambda+\mu}}{\Gamma(\mu)}\int_0^1 u^{\sigma}(1-u)^{\mu-1}\Big(\int_0^\infty p^{r-1}$$
$$\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-p}{u}\Big)dp\Big)\Big(\int_0^\infty q^{s-1}$$
$$\times{}_1\Psi_1\Big(\sigma,\tau;\frac{-q}{1-u}\Big)dq\Big)du,$$
$$=\frac{z^{\lambda+\mu}}{\Gamma(\mu)}\int_0^1 u^{\lambda}(1-u)^{\mu-1}du$$
$$\times\Big(\int_0^\infty u^{r-1}v^{r-1}w^{s-1}$$
$$\times{}_1\Psi_1(\sigma,\tau;-v)udv\Big)$$
$$\times\Big(\int_0^\infty(1-u)^{s-1}{}_1\Psi_1(\sigma,\tau;-w)$$
$$\times(1-u)dw\Big)$$
$$=\frac{z^{\lambda+\mu}}{\Gamma(\mu)}\mathcal{B}(\lambda+r+1,s+\mu)^{\psi}\Gamma^{(\sigma,\tau)}(r)$$
$$\times\Gamma^{(\sigma,\tau)}(s).$$

Theorem 4.5. *Let* $\mathcal{R}(s)>\mathcal{R}(\mu)>0,\mathcal{R}(b)>-1,\tau\in R,\sigma\in(0,2),\tau>1,$ *then*

$$\mathfrak{M}[{}^{\psi}I^{\mu,\sigma,\tau}_{z,p}\{(1-z)^{-b}\};p\to r,q\to s]$$
$$=\frac{{}^{\psi}\Gamma^{(\sigma,\tau)}(r)\,{}^{\psi}\Gamma^{(\sigma,\tau)}(s)z^{\mu}}{\Gamma(\mu)\mathcal{B}(r+1,s+\mu)}\mathcal{F}(b,r+1;r+s+\mu+1,z). \tag{46}$$

Proof. We use the series expansion of $(1-z)^{-b}=\sum_{k=0}^{\infty}\frac{(b)_k}{k!}z^k$ in left hand side of equation (46), then we get

$$\mathfrak{M}[{}^{\psi}I^{\mu,\sigma,\tau}_{z,p}\{(1-z)^{-b}\};p\to r,q\to s]$$
$$=\sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathfrak{M}[{}^{\psi}I^{\mu,\sigma,\tau}_{z,p}\{z^k\};p\to r,q\to s],$$
$$=\frac{{}^{\psi}\Gamma^{(\sigma,\tau)}(r)\,{}^{\psi}\Gamma^{(\sigma,\tau)}(s)z^{\mu}}{\Gamma(\mu)}\sum_{k=0}^{\infty}\frac{(b)_k\,z^k}{k!}$$
$$\times\mathcal{B}(k+r+1,s+\mu)$$
$$=\frac{{}^{\psi}\Gamma^{(\sigma,\tau)}(r)\,{}^{\psi}\Gamma^{(\sigma,\tau)}(s)z^{\mu}}{\Gamma(\mu)\mathcal{B}(r+1,s+\mu)}\mathcal{F}(b,r+1;r+s+\mu+1,z).$$

Hence the proof.

Conclusions

In the present study, we developed a new generalized beta function and the some of it's important results. Apart of this we also established generalized fractional integral operator containing this new generalized beta function in its kernel and evaluated some properties. The result shown in this paper are general in nature but can be extended to establish other properties of special functions. These extensions are useful in numerous research fields such as engineering, chemical, and physical problems and statistical distribution theory.

References

Agarwal, P., Nieto, J.J., Luo, M.J.(2017) Extended Riemann–Liouville type fractional derivative operator with applications. *Appl Math Inf Sci* **15**(1): *12–29.*

Ata, E.(2018) Generalized Beta Function Defined by Wright function. *arXiv*:1803.03121v1 [math.CA].

Atash, A.A.,Barahmah, S.S.,Kulib, M.A.(2018) On a new extensions of extended gamma and beta functions. *Int J Stat Appl Math* **3**(6):14-18.

Baleanu, D., Agarwal, P., Parmar, R.K., Alqurashi, M.M. Salahshour, S. (2015) Extended Riemann Liouville fractional derivative operator and its applications. *J Nonlinear Sci Appl* **10** : 2914–2924.

Bohner, M., Rahman, G.,Mubeen, S., Nisar, K. S. (2018) A further extension of the extended Riemann –Liouville fractional derivative operator. *Turk J Math* **42**:2631–2642.

Chaudhry, M.A., Qadir, A., Rafique, M., Zubair, S.M.(1997) Extension of Euler's beta function. *J Comput Appl Math* **78**:19–32.

Chaudhry, M.A., Qadir, A., Srivastava, H. M., Paris, R. B. (2004) Extended hypergeometric and confiuent hypergeometric functions. *Appl Math Comput* **159** : 589-602.

Choi, J., Rathie, A. K., Parmar, R. K.(2014) Extension of extended beta, hypergeometric and confluent hypergeometric functions. *Honam Math J* **36**(2):357-385.

Mubeen, S., Rahman, G., Nisar, K.S., Choi, J., Arshad, M.(2017)An extended beta function and its properties. *Far East J Math Sci* **102**:1545-1557.

Özergin, E., Özarslan, M.A., Altin, A.(2011) Extension of gamma,beta and hypergeometric functions. *J Comput Appl Math* **235**: 4601–4610.

Özarslan, M.A., Özergin, E.(2010) Some generating relations for extended hypergeometric functions via generalized fractional derivative operator. *Math Comput Model* **52:**1825–1833.

Parmar, R.K, Chopra, P., Paris, R.B.(2017) On an extension of extended beta and hypergeometric functions. *J Class Anal* **11:** 91–106.

Rahman, G., Mubeen, S., Nisar, K.S., Choi, J. (2019) Certain extended special functions and Fractional Integral and Derivative Operators via an Extended Beta Function.*Nonlinear Funct Anal Appl* **24**(1):1-13.

Shadab, M., Faisal, M., Luis, J., Bonilla, L.(2018) A new Riemann–Liouville type fractional derivative operator and its application in generating functions. *Adv Differ Equ* **2018**:167.

Srivastava, H. M., Agarwal, P.(2013) Certain fractional integral operators and the generalized incomplete hypergeometric functions. *Appl Appl Math* **8**(2):333–345.